九十七學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

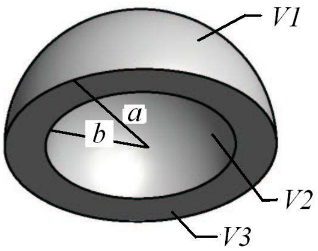
[Griffiths Ch.1-3] 2008/11/18, 10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

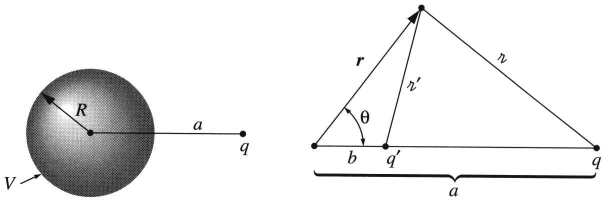
$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} \quad \text{and} \quad \nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$$

1. (6%, 7%, 7%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a,\theta)=0$, $V_2(b,\theta)=V_0(2\cos\theta - 5\cos\theta\sin^2\theta)$, $V_3(r,\pi/2)=0$. V_0 is a constant.
   (a) Show the general solution in the region $b \le r \le a$.
   (b) Determine the potential in the region $b \le r \le a$, using the boundary conditions.
   (c) Calculate the electric field in the region $b \le r \le a$.
   [Hint: $P_0(x)=1,\ P_1(x)=x,\ P_2(x)=(3x^2-1)/2,$ and $P_3(x)=(5x^3-3x)/2$.]

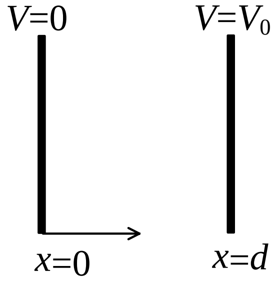


2. (10%, 10%) A point charge q is situated a large distance $\mathbf{r}$ from a neutral atom of polarizability α.
   (a) Find the induced dipole moment of the atom $\mathbf{p}$.
   (b) Find the force between them (attractive or repulsive).

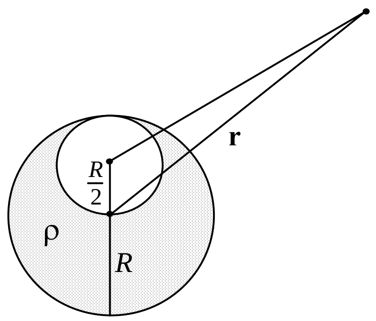
3. (10%, 10%) A point charge q is situated at distance a from the center of a conducting sphere of radius R. The sphere is maintained at the constant potential V_0.
   (a) Find the position and the value of the image charge.
   (b) Verify that the tangential component of the electric field is zero throughout on the surface of the metal sphere.
   [Hint: 1. use the notations shown below. 2. Assume q lays on the z-axis]



4. (8%, 6%, 6%) Consider two infinite parallel metal plates separated by a distance *s* are at potential 0 and V_0 as shown in the figure. .
   (a) Use Poisson's equation to find the potential *V* in the region between the plates where the space change density is $\rho = \rho_0 \dfrac{x}{d}$.
   (b) Find the electric field **E** in the region between the plates.
   (c) Use the boundary condition to determine the charge densities on the plates.
   [Note: The electric field inside the metal plate is zero $E_{inside} = 0$.]



5. (8%, 6%, 6%) Consider a hollowed charged sphere with radius *R* and uniform charge density ρ as shown in the figure. The inner radius of the spherical cavity is $R/2$.
   (a) If the observer is very far from the charged sphere, find the multiple expansion of the potential *V* in power of $1/r$
   (b) Find the dipole moment **p**.
   (c) Find the electric field **E** up to the dipole term.
   [Note: Specify a vector with both magnitude and direction.]



1.

(a)

$$\text{Boundary condition } \begin{cases} \text{(i) } V_1(a,\theta)=0 \\ \text{(ii) } V_2(b,\theta)=V_0(2\cos\theta-5\cos\theta\sin^2\theta)=V_0(5\cos^3\theta-3\cos\theta)=2V_0P_3 \\ \text{(ii) } V_3(r,\theta=\pi/2)=0 \end{cases}$$

General solution $V(r,\theta)=\sum_{\ell=0}^{\infty}(A_\ell r^\ell+B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

(b)

B.C. (i) $\rightarrow V(a,\theta)=\sum_{\ell=0}^{\infty}(A_\ell a^\ell+B_\ell a^{-(\ell+1)})P_\ell(\cos\theta)=0 \quad \Rightarrow B_\ell=-A_\ell a^{2\ell+1}$

B.C. (ii) $\rightarrow V(b,\theta)=\sum_{\ell=0}^{\infty}(A_\ell b^\ell+B_\ell b^{-(\ell+1)})P_\ell(\cos\theta)=2V_0P_3(\cos\theta)$

Comparing the coefficiency $\Rightarrow A_3b^3+B_3b^{-4}=2V_0,\ A_\ell=B_\ell=0$ for $\ell=0,1,2,4,5,...$

B.C. (iii) $\rightarrow V(r,\theta=\frac{\pi}{2})=(A_3r^3+B_3r^{-4})P_3(0)=0$

$\Rightarrow A_\ell=B_\ell=0$ except $\ell=3$,

$$A_3=\frac{2V_0b^4}{b^7-a^7} \text{ and } B_3=-\frac{2V_0b^4a^7}{b^7-a^7}$$

$$\therefore V(r,\theta)=\left(\frac{2V_0}{b^7-a^7}b^4r^3-\frac{2V_0}{b^7-a^7}b^4a^7r^{-4}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)$$

(c)

$$V(r,\theta)=\left(\frac{2V_0}{b^7-a^7}b^4r^3-\frac{2V_0}{b^7-a^7}b^4a^7r^{-4}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)$$

$$\begin{aligned}\mathbf{E}=-\nabla V&=-\frac{\partial V}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}\\ &=-\left(\frac{6V_0}{b^7-a^7}b^4r^2-\frac{8V_0}{b^7-a^7}b^4a^7r^{-5}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)\hat{\mathbf{r}}\\ &\quad+\left(\frac{6V_0}{b^7-a^7}b^4r^2-\frac{8V_0}{b^7-a^7}b^4a^7r^{-5}\right)\left(\frac{15\cos^2\theta\sin\theta-3\sin\theta}{2}\right)\hat{\boldsymbol{\theta}}\end{aligned}$$

2.

(a)

The electric field of a point charge $\mathbf{E}=\frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{\mathbf{r}}$

The induced dipole moment $\mathbf{p}=\alpha\mathbf{E}=\frac{1}{4\pi\varepsilon}\frac{\alpha q}{r^2}\hat{\mathbf{r}}$

(b)

The total electric static energy $U=-(\frac{1}{2})\mathbf{p}\cdot\mathbf{E}=-(\frac{1}{2})(\frac{1}{4\pi\varepsilon})^2\frac{\alpha q^2}{r^4}$

Note: $(\frac{1}{2})$ comes from the fact that the dipole moment $\mathbf{p}$ is induced by $\mathbf{E}$.

The force bewteen is $\mathbf{F}=-\nabla U=\ -2\alpha(\frac{1}{4\pi\varepsilon})^2\frac{q^2}{r^5}\hat{\mathbf{r}}$ attractive.

The direction of the induced dipole $\mathbf{p}$ is in line with the electric field $\mathbf{E}$ generated by the charge q.

3.(a)

Assume the image charge q' is placed at a distance b from the center of the sphere.
It is equipotential on the surface of a grounded sphere.
Using two boundary conditions at P_1 and P_2

$$\left.\begin{aligned}&\text{At } P_1\text{: } \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R-b}+\frac{q}{a-R}\right)=0\\ &\text{At } P_2\text{: } \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R+b}+\frac{q}{a+R}\right)=0\end{aligned}\right\}\ \text{two equations and two unknowns } (q' \text{ and } b)$$

$b=\frac{R^2}{a}$(position), $\quad q'=-\frac{R}{a}q$(value of the image charge)

(b)

The potential outside the sphere when V=0

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\left\{\frac{q'}{|\mathbf{r}-b\hat{\mathbf{z}}|}+\frac{q}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right\},\ \text{where } b=\frac{R^2}{a}\ \text{and } q'=-\frac{R}{a}q$$

The potential outside the sphere when $V=V_0$

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\left\{\frac{4\pi\varepsilon_0 RV_0}{r}+\frac{q'}{|\mathbf{r}-b\hat{\mathbf{z}}|}+\frac{q}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right\}$$

$$\text{where}\ \begin{cases}|\mathbf{r}-b\hat{\mathbf{z}}|=\sqrt{(r^2\sin^2\theta+(r\cos\theta-b)^2}=\sqrt{(r^2-2br\cos\theta+b^2)}\\ |\mathbf{r}-a\hat{\mathbf{z}}|=\sqrt{(r^2\sin^2\theta+(r\cos\theta-a)^2}=\sqrt{(r^2-2ar\cos\theta+a^2)}\end{cases},$$

On the surface of the metal sphere, $\mathbf{E}=-\nabla V(\mathbf{r})=-\frac{\partial V}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}$

$$\begin{aligned}E_\theta&=-\frac{1}{r}\frac{\partial V}{\partial\theta}=-\frac{1}{4\pi\varepsilon_0}\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{q'}{\sqrt{(r^2-2br\cos\theta+b^2)}}+\frac{q}{\sqrt{(r^2-2ar\cos\theta+a^2)}}\right\}\\ &=-\frac{1}{4\pi\varepsilon_0}\frac{1}{r}\left(\frac{q'(-2br\sin\theta)}{(r^2-2br\cos\theta+b^2)^{3/2}}+\frac{q(-2ar\sin\theta)}{(r^2-2ar\cos\theta+a^2)^{3/2}}\right)\\ &=\frac{2\sin\theta}{4\pi\varepsilon_0}\left(\frac{q'b}{(r^2-2br\cos\theta+b^2)^{3/2}}+\frac{qa}{(r^2-2ar\cos\theta+a^2)^{3/2}}\right)\end{aligned}$$

$$\frac{q'b}{(R^2+2bR\cos\theta-b^2)^{3/2}}=\frac{-\frac{R}{a}q\frac{R^2}{a}}{(R^2-2\frac{R^2}{a}R\cos\theta+\frac{R^4}{a^2})^{3/2}}=\frac{-\frac{R^3}{a^2}q}{\frac{R^3}{a^3}(a^2-2aR\cos\theta+R^2)^{3/2}}$$

$$=-\frac{qa}{(R^2-2aR\cos\theta+a^2)^{3/2}}$$

$$E_\theta(@\,r=R)=\frac{2\sin\theta}{4\pi\varepsilon_0}\left(-\frac{qa}{(R^2-2aR\cos\theta+a^2)^{3/2}}+\frac{qa}{(R^2-2aR\cos\theta+a^2)^{3/2}}\right)=0$$

4.(a)

$$\frac{d^2V}{dx^2}=-\frac{\rho_0 x}{\varepsilon_0 d}\quad\Rightarrow\quad V(x)=-\frac{\rho_0 x^3}{6\varepsilon_0 d}+c_1x+c_2$$

Use the boundary conditions to determine the coefficiencies.

$$\begin{cases} V(0)=0\quad\Rightarrow\quad c_2=0\\ V(d)=V_0=-\frac{\rho_0 d^2}{6\varepsilon_0}+c_1 d\quad\Rightarrow\quad c_1=\frac{V_0}{d}+\frac{\rho_0 d}{6\varepsilon_0}\end{cases}$$

$$V(x)=-\frac{\rho_0 x^3}{6\varepsilon_0 d}+(\frac{V_0}{d}+\frac{\rho_0 d}{6\varepsilon_0})x$$

(b)

$$V(x)=-\frac{\rho_0 x^3}{6\varepsilon_0 d}+(\frac{V_0}{d}+\frac{\rho_0 d}{6\varepsilon_0})x$$

$$\mathbf{E}=-\nabla V(x)=\frac{\rho_0 x^2}{2\varepsilon_0 d}-(\frac{V_0}{d}+\frac{\rho_0 d}{6\varepsilon_0})\hat{\mathbf{x}}$$

(c)

$$\begin{cases}\mathbf{E}_{x=0}=-(\frac{V_0}{d}+\frac{\rho_0 d}{6\varepsilon_0})\hat{\mathbf{x}}\\ \mathbf{E}_{x=d}=(\frac{2\rho_0 d}{3\varepsilon_0}-\frac{V_0}{d})\hat{\mathbf{x}}\end{cases}\quad\Rightarrow\quad\begin{cases}\sigma_{x=0}=\varepsilon_0(E_{outside}-\cancel{E_{inside}})=-(\frac{\varepsilon_0 V_0}{d}+\frac{\rho_0 d}{6})\\ \sigma_{x=d}=\varepsilon_0(E_{outside}-\cancel{E_{inside}})=(\frac{2\rho_0 d}{3}-\frac{\varepsilon_0 V_0}{d})\end{cases}$$

5.

(a)

Consider this problem as two charge spheres, one with charge density ρ the other with opposite charge density $-\rho$.

$$V_{big}=\frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}R^3)\quad\text{and}\quad V_{small}=\frac{1}{4\pi\varepsilon_0\left|\mathbf{r}-\frac{1}{2}\mathbf{R}\right|}(-\rho\frac{4\pi}{3}(\frac{R}{2})^3)$$

$$\frac{1}{\left|\mathbf{r}-\frac{1}{2}\mathbf{R}\right|}=\frac{1}{r}(1+(\frac{\frac{1}{2}R}{r})\cos\theta+\ldots)$$

Using the principle of superposition, we find,

$$V = \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}R^3) - \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}(\frac{R}{2})^3)(1+(\frac{\frac{1}{2}R}{r})\cos\theta+\ldots)$$

$$= \frac{1}{4\pi\varepsilon_0 r}\frac{7}{8}(\rho\frac{4\pi}{3}R^3) - \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}(\frac{R}{2})^3)(\frac{R}{2r})\cos\theta+\ldots, \quad \text{let } Q = \rho\frac{4\pi}{3}R^3$$

$$= \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}(\frac{Q}{8}\frac{R}{2})\cos\theta+\ldots$$

(b)

$$Q = \rho\frac{4\pi}{3}R^3$$

$$V = \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}(\frac{Q}{8}\frac{R}{2})\cos\theta+\ldots$$

The first term is the monopole term and the second term is the dipole term.

So the dipole moment $\mathbf{p} = -\frac{QR}{16}\hat{\mathbf{z}}$.

(c)

$$V = \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}(\frac{Q}{8}\frac{R}{2})\cos\theta+\ldots$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\varphi}}$$

$$= \left(\frac{1}{4\pi\varepsilon_0 r^2}\frac{7Q}{8} - \frac{2p}{4\pi\varepsilon_0 r^3}\cos\theta\right)\hat{\mathbf{r}} - \frac{p}{4\pi\varepsilon_0 r^3}\sin\theta\hat{\boldsymbol{\theta}}$$